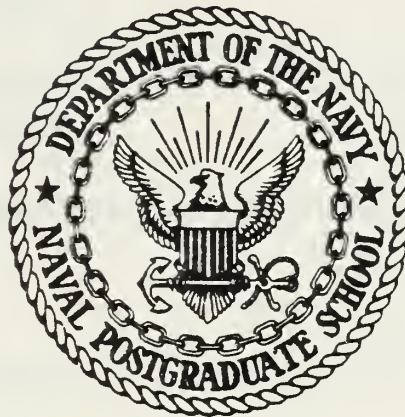


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THESIS

ARTIFICIAL DISPERSION OF AIM POINT
FOR ARTILLERY FIRE

by

Gwang Hee Ha

March 1983

Thesis Advisor:

J.G. Taylor

Approved for public release; distribution unlimited.

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Artificial Dispersion of Aim Point
for Artillery Fire

by

Gwang Hee Ha
Lieutenant Colonel, Korean Army
B.S., Republic of Korea Military Academy, 1970

Submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE IN OPERATIONS RESEARCH

from the

NAVAL POSTGRADUATE SCHOOL

March 1983

ABSTRACT

This thesis examines the problem of optimal aiming at an imperfectly located point target and an area target in which elements are uniformly or normally distributed. For an analytical simplicity, a one-dimensional target is considered and optimal aim points are determined when 2 rounds are fired.

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TABLE OF SYMBOLS AND ABBREVIATIONS

| SYMBOL | DEFINITION |
|---------------------|---|
| A_p | presented area |
| A_v | vulnerable area |
| $p(x,y)$ | average chance of kill for a cell |
| $\Delta x \Delta y$ | area of a cell |
| r | distance from point of burst |
| a | lethal range |
| $\ell(r)$ | lethality function |
| x_a | random variable of aim error on x-axis |
| μ_{xa} | mean of x_a |
| y_a | random variable of aim error on y-axis |
| μ_{ya} | mean of y_a |
| σ_{xa} | standard deviation of x_a |
| σ_{ya} | standard deviation of y_a |
| σ_a | standard deviation of x_a, y_a (used when $\sigma_{xa} = \sigma_{ya} = \sigma_a$) |
| x_d | random variable of impact on x-axis |
| y_d | random variable of impact on y-axis |
| σ_{xd} | standard deviation of x_d |
| σ_{yd} | standard deviation of y_d |
| σ_d | standard deviation of x_d, y_d (used when $\sigma_{xd} = \sigma_{yd} = \sigma_d$) |
| D_x | random variable of ballistic error on x-axis |
| D_y | random variable of ballistic error on y-axis |

| SYMBOL | DEFINITION |
|------------------------------|--|
| $g(x)$ | probability distribution of a target element |
| $P_{SSH} x_a$ | conditional hit probability of single shot with aiming realized at x_a in one dimension |
| $P_{SSH} (x_a, y_a)$ | conditional hit probability of single shot with aiming realized at (x_a, y_a) in two dimensions. |
| $P_{SSK} x_a$ | conditional kill probability of single shot with aiming realized at x_a in one dimension |
| $P_{SSK} (x_a, y_a)$ | conditional kill probability of single shot with aiming realized at (x_a, y_a) in two dimensions |
| $p_K(n)$ | salvo kill probability of a target with n rounds |
| X_ℓ | bias of true location of a target from origin |
| σ_ℓ | standard deviation of X_ℓ |
| $P_K (u_1, u_2, \dots, u_n)$ | kill probability of a target with n rounds aiming at (u_1, u_2, \dots, u_n) |
| σ_X | standard deviation of existence of elements |

I. INTRODUCTION

A. BACKGROUND

Artillery weapons are a key element of the combined-arms team for modern combat. Many studies and experiments have been conducted to determine the effective use of artillery weapons and have contributed to the establishment of employment doctrine for them. Such system-analysis work has included the development of quite complex, detailed Monte-Carlo simulations of artillery systems. Nevertheless, a simple combat model may yield a much clearer understanding of important relations that are difficult to perceive in a more complex model, and such insights can provide a valuable guidance for subsequent higher-resolution computerized investigations. Moreover, one can use such a simplified-auxiliary model for understanding the basic dynamics and behavior of a larger-scale complex-operational model.

This thesis follows this research strategy and uses a simple analytical model to investigate optimal aiming at an imperfectly located target. Both a point target and also an area target (in which target elements are uniformly or normally distributed) are considered. For simplicity's sake, one-dimensional models are considered. Therefore, the results obtained here should not be taken literally but should be interpreted as insights into optimal aiming in cases of more practical interest. Thus, there are still many unresolved

problems concerning optimal aiming and the optimal expenditure of ammunition (particularly in cases of uncertainty).

During the past twenty years, a large number of mathematical investigations have been carried out in two broad areas. First, descriptive studies have investigated the effectiveness of attacks on point and area targets by weapons having various systematic errors and destruction capabilities. Second, normative studies have investigated the optimal allocations of weapons against a group of point targets and for the defense of a group of point targets.

This thesis considers both these areas; it develops a descriptive model of artillery fire effectiveness (i.e., develops an expression for engagement-kill probability), and then it determines the optimal aim points (when two rounds are fired) to maximize fire effectiveness. Both point and area targets are considered. In this thesis, a point target is defined as a target whose size is small enough for it to be killed by a single round (i.e., the target is very small compared to the lethal area of a round). An area target is defined as a target whose size is too big for it to be killed by a single round (i.e., the target is much larger than the lethal area of a round).

B. VULNERABILITY CONSIDERATIONS

Vulnerability [1: pp. 15.1-15.15] of a target may be defined as the characteristics of a target, which describe its sensitivity to combat mechanisms. Therefore, the

vulnerability may be described as a function of the damage producing properties of the attacking weapon and the physical properties of the target. Thus, vulnerability involves considerations of a stochastic or probabilistic nature, which in turn will depend markedly on the conditions of both the attacking weapon and the target in the combat environment.

Conventionally, vulnerability is expressed in terms of an area of a volume for a given attack direction, for particular or specified conditions of the engagement of a weapon against a target. Because of the stochastic nature of this mechanism, it is necessary to describe the chance of damage on the target in probabilistic terms. Specifically, the probability of damage is determined by taking the vulnerable area of a target for given attack directions and dividing it by the presented area, which gives the conditional probability that a hit is a kill; i.e., $p(K|H)$.

The determination of vulnerable areas of a target may be described as follows.

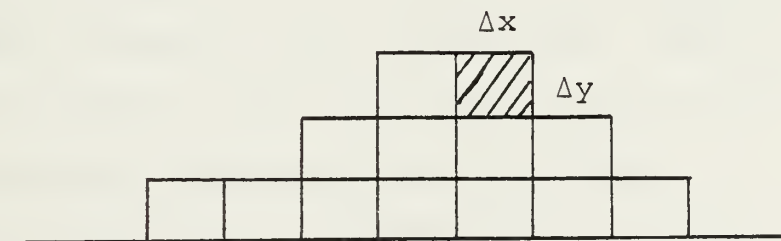


Fig. 1. Vulnerable Areas

$A_v = \sum_x \sum_y p(x,y) \Delta x \Delta y$, where $p(x,y)$ is the average chance of kill for a cell. Thus, A_v is a computed area obtained by

weighting suitable small areas by the conditional chance that a hit is a kill and summing those weighted cells over the whole presented area of the target.

$$p(K|H) = \frac{A_v}{A_p}.$$

This notion of conditional kill probability may be extended to the case where a target is killed by the shrapnel of artillery ammunition. It should be noted that determination of A_v is a long and detailed experimental process.

C. LETHALITY FUNCTIONS

Vulnerability is ordinarily a term used for the case where actual hits are obtained on a target such as tanks and aircraft. Lethality [1: pp. 15.1-15.15], on the other hand, refers primarily to the case where lethal or incapacitating fragments are projected over an area on the battlefield to incapacitate personnel. An artillery projectile is usually detonated on the ground or in the air, and consequently project lethal fragments over a large area. Such a projectile can still be effective even if it does not hit the target directly but merely detonates near the target.

In order to represent lethality in a functional form, the U.S. Army's Ballistic Research Laboratories have done extensive experimental work for a large number of weapon-target pairs. It has been observed experimentally, by counting perforations in wood panels placed at various distances from

the point of burst, that the number of fragments above a specified threshold tends to decrease as an exponential function of the distance from the point of burst. Since the shrapnel kills by its kinetic energy, this leads to a lethality function which is a negative exponential of the distances from the point of burst. Gaussian lethality function [2: pp. 9.1-9.6] is defined as follows:

$$\begin{aligned} \ell(r) &= p(\text{target killed} | \text{a round detonates at distance } r \text{ from the target}) \\ &= e^{-\frac{r^2}{2a^2}}, \end{aligned}$$

where a is a lethal range. It should be noted that this lethality function depends on both the weapon system and target type.

There appears to be no consistent or universally accepted way of defining lethal range, a . The following are commonly used.

$$a = \int_0^{\infty} \ell(r) dr$$

$$a = \sqrt{2/\pi} \int_0^{\infty} \ell(r) dr$$

$$a \text{ is solution to } \ell(r) = .5.$$

So, when one is given a number as a lethality, he must find out how it is defined. There is another type of lethality

function called 'cookie cutter' which is defined as follows.

$$l(r) = \begin{cases} P_0 & \text{for } 0 \leq r \leq R \\ 0 & \text{for } r > R \end{cases}$$

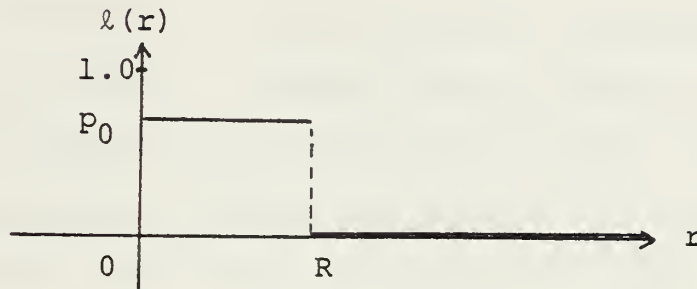


Fig. 2. Cookie Cutter Lethality Function

D. ERRORS AND PROBABILITY DISTRIBUTIONS

1. Pattern of Impacts [1: pp. 13.1-13.12]

In the firing of artillery weapons, the results are a two-dimensional pattern of impact points which exhibits an amount of scatter depending on the locating error of target, aiming error and ordinary ballistic dispersion. This two-dimensional pattern gives rise to various measures of dispersion and these measures of dispersion include sample standard deviation in each direction, extreme horizontal (vertical) dispersion, the mean horizontal (vertical) deviation, and the radial standard deviation, etc. It is important to note that the expected values of these various measures of dispersion depend on the sample size or number of rounds. That is, a group of several shots fired from a weapon represents a sample of rounds from a lot of ammunition, or a population

in statistical terms so that the scatter pattern will vary from one group to another. It is this random variation for us to model in order to estimate the probabilities of hitting a target.

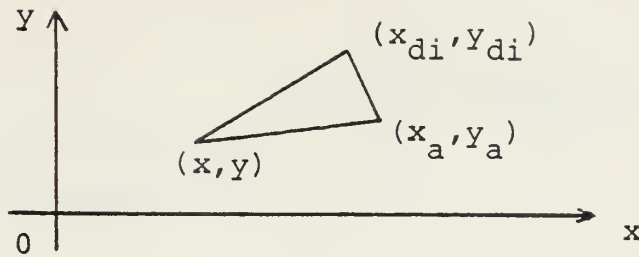
The round to round ballistic dispersion and the movement of the center of impact during firing a group of rounds affect the probability of hitting a target. As far as is known, the round to round ballistic dispersion remains relatively stable even though the center of impact may vary in some unpredictable manner. It has been observed that for the rifles fired at vertical targets pattern of impacts is nearly circular, whereas for the case of artillery weapons pattern of impacts is non-circular: i.e., dispersion in the range direction is considerably greater than that in deflection.

2. Probability Distribution for Errors [2: pp. 10.1-10.5]

We may think of the factors that affect the impact as being errors, and these errors, expressed as a misdistance, are considered to be the sum of the following random variables.

$$\text{misdistance} = (\text{Target location error} + \text{aim error}) + \text{ballistic error}$$

For convenience, all these errors are assumed to have probability distributions like, $p(x < X < x+dx) = f(x)dx$. Additionally, target location error may be combined with aim error and called again 'aim-error'. These errors can be described graphically as follows.



- Target element is located at (x, y)
- aiming is realized at (x_a, y_a)
- i th round lands at (x_{di}, y_{di})

Fig. 3. Error Description

a. Distribution of Aim Error

The aim error, a random variable denoted as (x_a, y_a) , is normally distributed with mean (μ_{xa}, μ_{ya}) and standard deviation $(\sigma_{xa}, \sigma_{ya})$.

Probability distribution for x_a :

$$f(x_a: \mu_{xa}, \sigma_{xa}) = \frac{1}{\sqrt{2\pi} \cdot \sigma_{xa}} e^{-\frac{1}{2} \left(\frac{x_a - \mu_{xa}}{\sigma_{xa}} \right)^2}$$

Probability distribution for y_a :

$$f(y_a: \mu_{ya}, \sigma_{ya}) = \frac{1}{\sqrt{2\pi} \cdot \sigma_{ya}} e^{-\frac{1}{2} \left(\frac{y_a - \mu_{ya}}{\sigma_{ya}} \right)^2}$$

Joint probability distribution for (x_a, y_a) :

$$f(x_a, y_a) = \frac{1}{2\pi\sigma_{xa}\sigma_{ya}\sqrt{1-\rho^2}} e^{-\frac{1}{2}\left[\left(\frac{x_a - \mu_{xa}}{\sigma_{xa}}\right)^2 + \left(\frac{y_a - \mu_{ya}}{\sigma_{ya}}\right)^2 - 2\rho \cdot \frac{x_a - \mu_{xa}}{\sigma_{xa}} \cdot \frac{y_a - \mu_{ya}}{\sigma_{ya}}\right]}$$

ρ is a correlation coefficient between x_a and y_a .

If it is assumed that $\rho = 0$ and $\sigma_{xa} = \sigma_{ya} = \sigma_a$, then the joint probability distribution will be simplified as follows.

$$f(x_a, y_a) = \frac{1}{2\pi\sigma_a^2} e^{-\frac{1}{2}\left[\left(\frac{x_a - \mu_{xa}}{\sigma_{xa}}\right)^2 + \left(\frac{y_a - \mu_{ya}}{\sigma_{ya}}\right)^2\right]}$$

b. Distribution of Ballistic Error

Components of ballistic error, D_x and D_y , may be defined as follows.

$$D_x = x_d - x_a$$

$$D_y = y_d - y_a$$

The ballistic error, a random variable denoted as (D_x, D_y) , is normally distributed with mean $(0,0)$ and standard deviation $(\sigma_{xa}, \sigma_{ya})$. Therefore, the random variables, x_d and y_d , are normally distributed with (x_a, y_a) and standard deviation

$(\sigma_{x_a}, \sigma_{y_a})$. The probability distribution for x_d is:

$$f(x_d: x_a, \sigma_{x_d}) = \frac{1}{\sqrt{2\pi}\sigma_{x_d}} e^{-\frac{1}{2}\left(\frac{x_d - x_a}{\sigma_{x_d}}\right)^2}$$

The probability distribution for y_d is:

$$f(y_d: y_a, \sigma_{y_d}) = \frac{1}{\sqrt{2\pi}\sigma_{y_d}} e^{-\frac{1}{2}\left(\frac{y_d - y_a}{\sigma_{y_d}}\right)^2}$$

The joint probability distribution for (x_d, y_d) is:

$$f(x_d, y_d) = \frac{1}{2\pi\sigma_{x_d}\sigma_{y_d}\sqrt{1-\rho^2}} e^{-\frac{1}{2}\left[\left(\frac{x_d - x_a}{\sigma_{x_d}}\right)^2 + \left(\frac{y_d - y_a}{\sigma_{y_d}}\right)^2 - 2\rho \cdot \frac{x_d - x_a}{\sigma_{x_d}} \cdot \frac{y_d - y_a}{\sigma_{y_d}}\right]}$$

ρ is a correlation coefficient between x_d and y_d . If it is assumed that $\rho = 0$, then the joint probability distribution will be simplified as follows.

$$f(x_d, y_d) = \frac{1}{2\pi\sigma_{x_d}\sigma_{y_d}} e^{-\frac{1}{2}\left[\left(\frac{x_d - x_a}{\sigma_{x_d}}\right)^2 + \left(\frac{y_d - y_a}{\sigma_{y_d}}\right)^2\right]}$$

E. TARGET DENSITY CONSIDERATION

Target density is defined as the probability distribution of elements in an area target.

$$p(\text{an element exists between } x \text{ and } x+dx) = g(x)dx$$

where $g(x)$ is the density function of an element. Target density is involved where it is desired to obtain the expected kill probability of an area target. This thesis examined two cases of target density: i.e., one for the uniformly distributed area target and the other for the normally distributed area target.

II. BASIC MODEL

A. HIT PROBABILITY MODEL

1. One Dimensional Model

The chance of hitting a target is, in large, dependent upon the aim error (with target location error) and the delivery error.

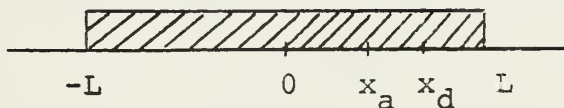


Fig. 4. One Dimensional Hit Pattern

Here the assumptions are:

1. A firer aims at the center of mass (origin) but he makes an aim error x_a .
2. A round lands at x_d .

$$p_{SSH}|x_a = p(\text{Hit} | \text{aiming realized at } x_a)$$

$$= \int_{-L}^L \frac{1}{\sqrt{2\pi}\sigma_d} e^{-\frac{1}{2}\left(\frac{x_d - x_a}{\sigma_d}\right)^2} dx_d$$

If there is no bias in aim, $x_a = 0$, and the standard deviation of ballistic error is large as compared to target size, small

target approximation will be as follows.

$$P_{SSH}|x_a \approx \frac{2L}{\sqrt{2\pi}\sigma_d}$$

2. Two Dimensional Model

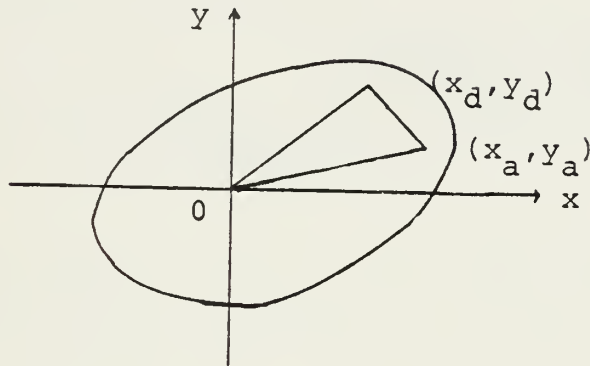


Fig. 5. Two Dimensional Hit Pattern

Using the same assumptions as in the one dimensional model, $P_{SSH}|(x_a, y_a)$ will be defined as follows.

$$P_{SSH}|(x_a, y_a) = p(\text{Hit} | \text{aiming realized at } (x_a, y_a))$$

$$= \iint_A \frac{1}{2\pi\sigma_{xd}\sigma_{yd}\sqrt{1-\rho^2}} e^{-\frac{1}{2}\left[\left(\frac{x_d-x_a}{\sigma_{xd}}\right)^2 + \left(\frac{y_d-y_a}{\sigma_{yd}}\right)^2 - 2\rho \cdot \frac{x_d-x_a}{\sigma_{xd}} \cdot \frac{y_d-y_a}{\sigma_{yd}}\right]} dx_d dy_d$$

Here, A is the area of a target and ρ is a correlation coefficient between x_d and y_d . If there is no bias in aim, $x_a = y_a = 0$, and $(\sigma_{xd}, \sigma_{yd})$ are large as compared to the size of a

target, and the small target approximation will be as follows.

$$P_{SSH}|(x_a, y_a) \approx \frac{A}{2\pi\sigma_{xd}\sigma_{yd}}.$$

For a rectangular target, $\rho = 0$.

$$P_{SSH}|(x_a, y_a) = \iint_A \frac{1}{2\pi\sigma_{xd}\sigma_{yd}} e^{-\frac{1}{2}\left[\left(\frac{x_d - x_a}{\sigma_{xd}}\right)^2 + \left(\frac{y_d - y_a}{\sigma_{yd}}\right)^2\right]} dx_d dy_d$$

For a circular target, $\rho = 0$, $x_a = y_a = 0$, and $\sigma_{xd} = \sigma_{yd} = \sigma_d$.

$$\begin{aligned} P_{SSH}|(x_a, y_a) &= \iint_A \frac{1}{2\pi\sigma_d^2} e^{-\frac{1}{2}\left(\frac{x_d^2 + y_d^2}{\sigma_d^2}\right)} dx_d dy_d \\ &= 1 - e^{-\frac{1}{2} \cdot \frac{R^2}{\sigma_d^2}}, \end{aligned}$$

where R is the radius of a circular target.

B. KILL PROBABILITY MODEL

1. One Dimensional Model

The chance of killing a point target is dependent upon the chance of a hit, the damage mechanism of the projectile and the vulnerability of a target. For developing a model, Gaussian lethality function is used instead of cookie cutter type lethality function, and this will lead to a more tractable analytical result.

$$\begin{aligned}
P_{SSK}|x_a &= \int_{-\infty}^{\infty} p(\text{target killed} | \text{a round lands between } x_d \text{ and } x_d+dx_d) \cdot p(\text{a round lands between } x_d \text{ and } x_d+dx_d) dx_d \\
&= \int_{-\infty}^{\infty} l(x_d) \cdot f(x_d: x_a, \sigma_{xd}) dx_d = \frac{1}{\sqrt{2\pi}\sigma_d} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(\frac{x_d}{\sigma_d})^2 - \frac{1}{2}(\frac{x_d-x_a}{\sigma_d})^2} dx_d
\end{aligned}$$

Simplifying the above formula, the following result is obtained.

$$P_{SSK}|x_a [a: 9.2-9.4] = \frac{a}{\sqrt{a^2 + \sigma_d^2}} e^{-\frac{1}{2} \cdot \frac{x_a^2}{a^2 + \sigma_d^2}} \quad (1)$$

2. Two Dimensional Model

Two dimensional model is simply an extension of the one dimensional model. Assuming there is no interaction between x_d and y_d , $P_{SSK}|(x_a, y_a)$ will be as follows.

$$P_{SSK}|(x_a, y_a) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_{xd}\sigma_{yd}} e^{-\frac{1}{2}[(\frac{x_d-x_a}{\sigma_{xd}})^2 + (\frac{y_d-y_a}{\sigma_{yd}})^2] - \frac{1}{2} \cdot \frac{x_d^2 + y_d^2}{a^2}} dx_d dy_d$$

Simplifying the above formula, the following result is obtained.

$$P_{SSK}|(x_a, y_a) = \frac{a^2}{\sqrt{a^2 + \sigma_{xd}^2} \sqrt{a^2 + \sigma_{yd}^2}} e^{-\frac{1}{2}(\frac{x_a^2}{a^2 + \sigma_{xd}^2} + \frac{y_a^2}{a^2 + \sigma_{yd}^2})} \quad (2)$$

For the case of circular normal ballistic error,

$\sigma_{xd} = \sigma_{yd} = \sigma_d$, then

$$P_{SSK} | (x_a, y_a) = \frac{a^2}{a^2 + \sigma_d^2} e^{-\frac{1}{2} \cdot \frac{x_a^2 + y_a^2}{a^2 + \sigma_d^2}}$$

If there is no aim error, $x_a = y_a = 0$, then

$$P_{SSK} | (x_a, y_a) = \frac{a^2}{a^2 + \sigma_d^2} = P_{SSK} | x_a=0 \cdot P_{SSK} | y_a=0$$

It indicates that the two dimensional kill probability is just the multiplication of the one dimensional kill probabilities in this simple case.

III. SALVO FIRE MODEL (POINT TARGET)

A. INTRODUCTION

Sometimes, it is not possible to destroy or neutralize many targets with single round because hit probabilities may be too low, some targets maybe too large in size, relatively invulnerable, or lethality per round is often insufficient. In these cases where the kill probability per round is too low, several or many rounds must be fired sequentially at a target or with some pattern of aiming dispersions, if suitable damage is to be accomplished. In this chapter, kill probability models of salvo fire (firing n rounds sequentially at the same aim point) will be examined.

B. ONE DIMENSIONAL MODEL

Assumptions used in this model are:

1. Aim errors and delivery errors follow normal distributions as defined previously.
2. A firer engages a target with center of mass at the origin.
3. Use a common aim point for n rounds.
4. Use Gaussian lethality function.
5. Cumulative damage is negligible.
6. Sequential delivery errors are independent.

The salvo kill probability of n rounds is defined as follows:

$$\begin{aligned} P_{K|x_a}(n) &= p(\text{kill target with } n \text{ rounds} | \text{aiming realized at } x_a) \\ &= 1 - (1 - p_{SSK|x_a})^n \end{aligned}$$

Since $p_{SSK|x_a}$ was given in the previous chapter, unconditioning the above expression on aim point x_a gives the following result.

$$p_K(n) \text{ [2: 10.10-10.15]} = a \cdot \sum_{K=1}^n \binom{n}{K} \left(\frac{-a}{\sqrt{a^2 + \sigma_d^2}} \right)^{K-1} \cdot \frac{1}{\sqrt{a^2 + \sigma_d^2 + K\sigma_a^2}} \cdot \frac{1}{2} \cdot \frac{K\mu_a^2}{a^2 + \sigma_d^2 + K\sigma_a^2} \cdot e \quad (3)$$

Examination of this result indicates that $p_K(n)$ has maximum value when $\mu_a = 0$; i.e., when a firer makes no aim error, $p_K(n)$ has the following value.

$$p_K(n) = a \sum_{K=1}^n \binom{n}{K} \left(\frac{-a}{\sqrt{a^2 + \sigma_d^2}} \right)^{K-1} \cdot \frac{1}{\sqrt{a^2 + \sigma_d^2 + K\sigma_a^2}}$$

C. TWO DIMENSIONAL MODEL

It has been observed in the real world that observers tend to have a greater aim error in range than in deflection, and this greater range dispersion can also be seen in ballistic errors. Hence, it is desirable to develop a two dimensional model in which the probability distributions of random variables are elliptical normal; i.e., $\mu_{xa} \neq \mu_{ya}$, $\sigma_{xa} \neq \sigma_{ya}$, and $\sigma_{xd} \neq \sigma_{yd}$. The assumptions from the one dimensional model are used again, and additional assumptions are necessary.

1. There is no interaction between x_a and y_a .
2. Gaussian lethality function

$$l(r) = e^{-\frac{1}{2} \cdot \frac{(x_d - x)^2 + (y_d - y)^2}{a^2}}$$

where (x, y) is the location of target and it is the origin here.

$$\begin{aligned} p_{K| (x_a, y_a)}^{(n)} &= p(\text{kill target with } n \text{ rounds} | \text{aiming realized at } (x_a, y_a)) \\ &= 1 - (1 - p_{SSK| (x_a, y_a)})^n. \end{aligned}$$

Since $p_{SSK| (x_a, y_a)}$ was given as Equation (2), unconditioning the above formula on aim point (x_a, y_a) gives the following result.

$$p_K(n) = 1 - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - p_{SSK| (x_a, y_a)})^n \cdot f(x_a, y_a) dx_a dy_a$$

Substituting $p_{SSK| (x_a, y_a)}$ and simplifying the above expression leads to the following result.

$$\begin{aligned} p_K(n) &= \sum_{K=1}^n \binom{n}{K} (-1)^K \left(\frac{a^2}{\sqrt{a^2 + \sigma_{xd}^2} \sqrt{a^2 + \sigma_{yd}^2}} \right)^{K-1} \cdot a^2 \cdot \frac{1}{\sqrt{a^2 + \sigma_{xd}^2 + K\sigma_{xa}^2} \sqrt{a^2 + \sigma_{yd}^2 + K\sigma_{ya}^2}} \\ &\quad \cdot e^{-\frac{K}{2} \left(\frac{\mu_{xa}^2}{a^2 + \sigma_{xd}^2 + K\sigma_{xa}^2} + \frac{\mu_{ya}^2}{a^2 + \sigma_{yd}^2 + K\sigma_{ya}^2} \right)} \end{aligned} \quad (4)$$

$p_K(n)$ has its maximum value when $\mu_{xa} = \mu_{ya} = 0$, i.e.,

$$p_K(n) = \sum_{k=1}^n \binom{n}{k} (-1)^K \left(\frac{a^2}{\sqrt{a^2 + \sigma_{xd}^2} \sqrt{a^2 + \sigma_{yd}^2}} \right)^{K-1} \cdot a^2 \left(\frac{1}{\sqrt{a^2 + \sigma_{xd}^2 + K\sigma_{xa}^2} \sqrt{a^2 + \sigma_{yd}^2 + K\sigma_{ya}^2}} \right)$$

If $\sigma_{xd} = \sigma_{yd} = \sigma_d$ and $\mu_{xa} = \mu_{ya} = 0$ in (4), then

$$p_K(n) = \sum_{k=1}^n \binom{n}{k} (-1)^K \left(\frac{a^2}{a^2 + \sigma_d^2} \right)^{K-1} \cdot a^2 \cdot \frac{1}{a^2 + \sigma_d^2 + K\sigma_a^2}$$

This result is identical to the case where the ballistic error follows a circular normal distribution.

IV. OPTIMIZATION OF KILL PROBABILITY

A. INTRODUCTION

So far, probability models for single shot and salvo fire against a point target have been examined. It has been observed that these kill probabilities are dependent upon various parameters and assumptions, and the salvo kill probability is dependent upon the single shot kill probability. Therefore, it is attempted in this chapter to examine the relationships between parameters that maximize the single shot kill probability.

Since the two dimensional model is an extension of the one dimensional model, complicated to handle, and the result of the one dimensional model can be linked to the two dimensional case without much difficulty, the one dimensional model will be examined. It has been assumed that a firer knows the exact location of a target, but it is not always true in the real world. Before going into the case of the unknown target location, the case of the known location of a point target will be first examined.

B. KNOWN TARGET LOCATION

The single shot kill probability was defined as follows.

$$P_{SSK}|x_a = \frac{a}{\sqrt{a^2 + \sigma_d^2}} e^{-\frac{1}{2} \cdot \frac{x_a^2}{a^2 + \sigma_d^2}}$$

$$\frac{d}{dx_a} P_{SSK}|x_a = - \frac{x_a}{a^2 + \sigma_d^2} \cdot P_{SSK}|x_a$$

For some fixed value of a and σ_d , moving x_a (realized aim point) away from the target center of mass always reduces the kill probability.

$$\frac{d}{d\sigma_d} P_{SSK}|x_a = \frac{\sigma_d}{a^2 + \sigma_d^2} \left(\frac{x_a^2}{a^2 + \sigma_d^2} - 1 \right) \cdot P_{SSK}|x_a$$

For the first derivative to be positive, $\left(\frac{x_a^2}{a^2 + \sigma_d^2} - 1 \right) > 0$, i.e.,

$$|x_a| > \sqrt{a^2 + \sigma_d^2}$$

It is interesting to note that as σ_d gets bigger, there is some threshold in x_a beyond which the single shot kill probability increases.

C. UNKNOWN TARGET LOCATION

This is the case, for example, that a firer engages a target of which the center of mass is believed to be located at the origin but the true center of mass is located at x_ℓ , the distance from the origin.

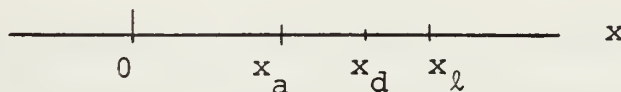


Fig. 6. Bias of Target Location

The same assumptions for the basic model are used again.

The lethality function needs to be modified as follows.

$$l(r) = e^{-\frac{1}{2} \left(\frac{x_d - x_\ell}{a} \right)^2}$$

Therefore, the single shot kill probability may be defined as follows.

$$p_{SSK|x_a} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_d} e^{-\frac{1}{2} \left(\frac{x_d - x_a}{\sigma_d} \right)^2} \cdot e^{-\frac{1}{2} \left(\frac{x_d - x_\ell}{a} \right)^2} dx_d$$

Evaluating this integral gives the following result.

$$p_{SSK|x_a} = \frac{a}{\sqrt{a^2 + \sigma_d^2}} e^{-\frac{1}{2} \cdot \frac{(x_a - x_\ell)^2}{a^2 + \sigma_d^2}}$$

This is the probability that a target located at x_ℓ will be killed by a single round of which the aiming point is realized at x_a .

$$\frac{d}{dx_a} p_{SSK|x_a} = - \frac{x_a - x_\ell}{a^2 + \sigma_d^2} \cdot p_{SSK|x_a}$$

For some fixed value of a and σ_d in this case, the single shot kill probability becomes maximum when a firer happens to aim at the true center of mass; i.e., as $(x_a - x_\ell)$ becomes larger,

the kill probability decreases. There comes up a question of how a firer can get x_a close to the true center of mass, x_ℓ , in a situation that there exists an uncertainty in the location of a target; i.e., x_ℓ is unknown. There seems to be no way of shortening $(x_a - x_\ell)$ with a single round. Thus it is clear that one can not expect the desired effect by hit or close hit with a single round.

One way to solve this problem would be the adjusting fire [1: pp. 14-3]. By adjusting fire it is easy to make x_a close to x_ℓ . It can also be shown that adjusting fire is more effective than salvo fire. Salvo kill probability for this point target of unknown location will be summarized as follows:

$$F_K(n) = \sum_{K=1}^n \binom{n}{K} (-1)^K \left(\frac{a}{\sqrt{a^2 + \sigma_d^2}} \right)^{K-1} \cdot a \cdot \frac{1}{\sqrt{a^2 + \sigma_d^2 + K\sigma_d^2}} e^{-\frac{1}{2} \frac{K(x_\ell - \mu_a)^2}{a^2 + \sigma_d^2 + K\sigma_d^2}}$$

Here, the term $(x_\ell - \mu_a)$ is fixed during the entire period of salvo fire, whereas it is decreased for each successive round in adjusting fire.

Another way to solve this problem would be the pattern fire with artificial dispersion of aiming points, but this method still raises the question of how far those aiming points should be separated from one another. These two suggestions (adjusting fire and artificial dispersion of aiming points

for a point target of unknown location) may deserve another area of investigation. The latter will be investigated in a simple case where $n = 2$ in the next chapter.

It is noted that this model of unknown location may be viewed in a different way with a slight modification of assumptions used. If we assume that the origin is true center of mass of an area target with a specified size, $-L$ to L , and x_ℓ is simply the location of an element, then $p_{SSK|x_a}$ becomes the probability that an element located at x_ℓ will be killed by a single round of which aiming point is realized at x_a . This concept may be shown in Fig. 7.

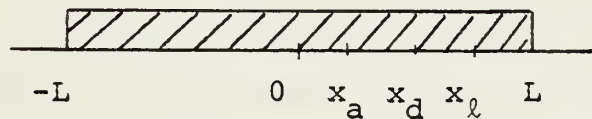


Fig. 7. Extension of Concept of Unknown Point Target to an Area Target

V. ARTIFICIAL DISPERSION OF AIMING POINTS

A. INTRODUCTION

There have been many attempts to improve the effectiveness of artillery fire by dispersing aiming points artificially. However, finding an optimal dispersion is really a complex problem for which a general solution is not yet available for all cases of interest. As an extension to the last chapter, the case of an imperfectly located point target will be examined and then the case of an area target in which elements are uniformly or normally distributed will be examined. One dimensional models will be investigated. For a general solution, n optimal aiming points must be obtained but it is almost impossible analytically. So, the solution is restricted to the case where $n = 2$.

B. IMPERFECTLY LOCATED POINT TARGET [3]

Assumptions are as follows.

1. A firer engages a target with unknown precise location

$$f(x_\ell; 0, \sigma_\ell) = \frac{1}{\sqrt{2\pi}\sigma_\ell} e^{-\frac{1}{2}\left(\frac{x_\ell}{\sigma_\ell}\right)^2}$$

2. n rounds are fired at distinct aiming points

$$(u_1, u_2, \dots, u_n)$$

3. distribution of delivery error for the i -th round is:

$$f(x_d; u_i, \sigma_d) = \frac{1}{\sqrt{2\pi}\sigma_d} e^{-\frac{1}{2}\left(\frac{x_d - u_i}{\sigma_d}\right)^2}$$

4. lethality function is:

$$l(r) = e^{-\frac{1}{2}\left(\frac{x_d - x_\ell}{a}\right)^2}$$

5. cumulative target damage is negligible.

$$\begin{aligned} p_{SSK|x_\ell, u_i} &= p(\text{kill a target} \mid \text{aim at } u_i \text{ for,} \\ &\quad \text{at } x_\ell \mid \text{i-th round}) \\ &= \frac{a}{\sqrt{a^2 + \sigma_d^2}} e^{-\frac{1}{2} \cdot \frac{(u_i - x_\ell)^2}{a^2 + \sigma_d^2}} \end{aligned}$$

For n rounds, aimed at (u_1, u_2, \dots, u_n) ,

$$\begin{aligned} p_K|x_\ell, (u_1, u_2, \dots, u_n) &= 1 - \prod_{i=1}^n (1 - p_{SSK|x_\ell, u_i}) \\ &= 1 - \prod_{i=1}^n \left(1 - \frac{a}{\sqrt{a^2 + \sigma_d^2}} e^{-\frac{1}{2} \cdot \frac{(u_i - x_\ell)^2}{a^2 + \sigma_d^2}}\right) \end{aligned}$$

Unconditioning of $p_K|(u_1, u_2, \dots, u_n)$ over x_ℓ results in:

$$p_K|(u_1, u_2, \dots, u_n) = 1 - \frac{1}{\sqrt{2\pi}\sigma_\ell} \int_{-\infty}^{\infty} \left[\prod_{i=1}^n \left(1 - \frac{a}{\sqrt{a^2 + \sigma_d^2}} e^{-\frac{1}{2} \cdot \frac{(u_i - x_\ell)^2}{a^2 + \sigma_d^2}}\right) \right] e^{-\frac{1}{2}\left(\frac{x_\ell}{\sigma_\ell}\right)^2} dx_\ell$$

Assuming $n = 2$ and $u_1 = -u_2 = |u|$

$$p_K|_{u_1, u_2} = p_K(u) = \frac{2a}{\sqrt{a^2 + \sigma_\ell^2 + \sigma_d^2}} e^{-\frac{1}{2} \frac{u^2}{a^2 + \sigma_\ell^2 + \sigma_d^2}} - \frac{a^2}{\sqrt{a^2 + \sigma_d^2} \sqrt{a^2 + 2\sigma_\ell^2 + \sigma_d^2}} e^{-\frac{u^2}{a^2 + \sigma_d^2}}$$

The problem is to find u that maximizes $p_K(u)$.

$$p_K(u) = p_K(-u)$$

$$\frac{d}{du} p_K(u) = \frac{-2au}{(a^2 + \sigma_\ell^2 + \sigma_d^2)^{3/2}} e^{-\frac{1}{2} \frac{u^2}{a^2 + \sigma_\ell^2 + \sigma_d^2}} + \frac{2a^2 u}{(a^2 + \sigma_d^2)^{3/2} \sqrt{a^2 + \sigma_d^2 + 2\sigma_\ell^2}} e^{-\frac{u^2}{a^2 + \sigma_d^2}}$$

$p_K(u)$ is $\begin{pmatrix} \text{increasing} \\ \text{extreme} \\ \text{decreasing} \end{pmatrix}$ at u , when $p'_K(u) \begin{pmatrix} > \\ = \\ < \end{pmatrix} 0$.

By putting $A = \frac{a^2 + \sigma_d^2}{\sigma_\ell^2}$, $B = \frac{\sigma_d^2}{a^2}$, and $v = \frac{u}{\sigma_d}$, inequality can be seen in an easy way.

$p_K(u)$ is $\begin{pmatrix} \text{increasing} \\ \text{extreme} \\ \text{decreasing} \end{pmatrix}$ at u , when $v^2 \begin{pmatrix} < \\ = \\ > \end{pmatrix} \frac{2(1+B)(1+A)}{(A+2)B} \ln \frac{(A+1)^{3/2}}{\sqrt{B+1}A\sqrt{A+2}}$

Let v increase from 0 to ∞ . If inside of the logarithm is less than one, then $p_K(u)$ always decreases as u increases.

$p'_K(u) = 0$, when $u = 0$, or

$$v^2 = \frac{2(1+B)(1+A)}{(A+2)B} \ln \frac{(A+1)^{3/2}}{\sqrt{B+1}A\sqrt{A+2}}$$

In this case the optimal will be at $u = 0$, i.e., it is better not to disperse aiming points.

If the inside of the logarithm is greater than one, then we have first $<$, $=$, and finally $>$ in the above inequalities. That is, $p_K(u)$ is first increasing, at extreme, and then decreasing. In other words, $p_K(u)$ is at a maximum when the relationship $=$ holds. So, the optimal dispersion of two rounds for an imperfectly located point target will be at the following points.

$$|u| = \sigma_{cl} \left(\frac{(1+B)(1+A)}{(A+2)B} \ln \frac{(A+1)^3}{(B+1)A^2(A+2)} \right)^{1/2}$$

These results suggest that when one doesn't know the location of a target precisely, bracketing a target would do better.

C. AREA TARGET

1. General Model Description

Examination of an area target involves additional considerations of the distribution of elements and the size of a target. Before going into the specific cases of uniform or normal distribution, a model of an area target will be developed.

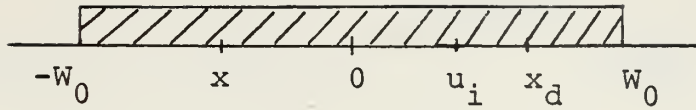


Fig. 8. One Dimensional Area Target Description

Additional assumptions to those used in the previous section are:

1. x is the location of an element of an area target (known location)
2. the center of mass is the origin
3. $-W_0$ to W_0 is the size of a target
4. $g(x)$ is the probability distribution of elements.

$$P_{SSK|x, x_d, u_i} = p(\text{an element at } x | \text{a round detonates at } x_d) \\ \text{will be killed with aiming at } u_i$$

$$= \frac{1}{a} (x - x_d) = e^{-\frac{1}{2} \left(\frac{x - x_d}{a} \right)^2}$$

$$P_{SSK|x_d, u_i} = p(\text{an area target} | \text{a round detonates at } x_d) \\ \text{will be killed with aiming at } u_i$$

$$= \int_{-W_0}^{W_0} g(x) \cdot e^{-\frac{1}{2} \left(\frac{x - x_d}{a} \right)^2} dx$$

$$P_{SSK|u_i} = p(\text{area target} | \text{aiming at } u_i) \\ \text{will be killed}$$

$$P_{SSK|u_i} = \int_{-\infty}^{\infty} \int_{-W_0}^{W_0} g(x) e^{-\frac{1}{2}\left(\frac{x-x_d}{a}\right)^2} \cdot \frac{1}{\sqrt{2\pi}\sigma_d} e^{-\frac{1}{2}\left(\frac{x_d-u_i}{\sigma_d}\right)^2} dx dx_d$$

Manipulating inside of the double integral,

$$\begin{aligned} P_{SSK|u_i} &= \int_{-W_0}^{W_0} g(x) \left[\frac{1}{\sqrt{2\pi}\sigma_d} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-x_d}{a}\right)^2} e^{-\frac{1}{2}\left(\frac{x_d-u_i}{\sigma_d}\right)^2} dx_d \right] dx \\ &= \int_{-W_0}^{W_0} g(x) \cdot P_{SSK|x, u_i} dx \end{aligned}$$

$$\begin{aligned} P_{SSK|x, u_i} &= P(\text{element at } x \text{ will be killed} | \text{aiming at } u_i) \\ &= \frac{a}{\sqrt{a^2 + \sigma_d^2}} e^{-\frac{1}{2} \cdot \frac{(x-u_i)^2}{a^2 + \sigma_d^2}} \end{aligned}$$

This conditional probability is the same as that developed in the case of an imperfectly located target. Thus,

$$P_{SSK|u_i} = \int_{-W_0}^{W_0} g(x) \cdot \frac{a}{\sqrt{a^2 + \sigma_d^2}} e^{-\frac{1}{2} \cdot \frac{(x-u_i)^2}{a^2 + \sigma_d^2}} dx \quad (5)$$

$$\text{For } n = 2, P_{K|u_1, u_2} = 1 - \prod_{i=1}^2 (1 - P_{SSK|u_i}) \quad (6)$$

Simplification of equation (5) is not easy because of the sharply defined area, $-W_0$ to W_0 . Thus, an approximation procedure will be introduced.

2. Diffuse Target Approximation

By replacing sharply defined area target by a diffuse target, evaluation of the integral in (5) becomes easy. It doesn't cause any great differences in the result. As to the function which expresses this continuous variability of the effect on a diffuse target, it is proposed to use a Gaussian distribution, e^{-x^2/W^2} .

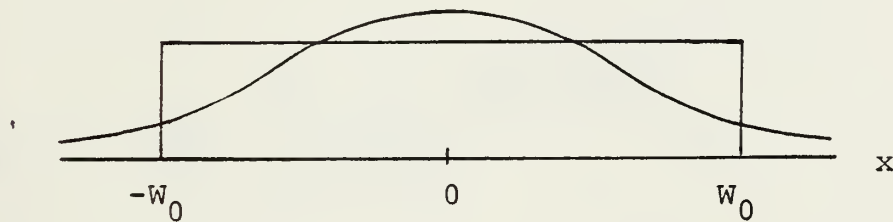


Fig. 9. Diffuse Target Approximation

This modification can be supported by the following arguments. Discrepancies in two areas can further be decreased by adjusting W so that the two areas be equal.

$$\int_{-W_0}^{W_0} dx = \int_{-\infty}^{\infty} e^{-x^2/W^2} dx$$

$$2W_0 = W \cdot \sqrt{\pi} \qquad W = \frac{2}{\sqrt{\pi}} W_0 = 1.13 W_0$$

This numerical result (i.e., nearness of the factor 1.13 to 1) indicates that such an adjustment of W is not likely to cause a great change and the approximation need not at all be more remote from reality than the original situation if further possibilities are taken into consideration. For example, if the width W_0 of a target is not precisely known, if W_0 is not sharply defined because of irregular shape of the target, etc.

3. Uniform Distribution of Target Elements

In this, the distribution of the elements, $g(x) = \frac{1}{2W_0}$

$$\begin{aligned}
 P_{SSK|u_i} &= \frac{1}{2W_0} \int_{-W_0}^{W_0} \frac{a}{\sqrt{a^2 + \sigma_d^2}} e^{-\frac{1}{2} \cdot \frac{(x-u_i)^2}{a^2 + \sigma_d^2}} dx \\
 &= \frac{1}{2W_0} \int_{-\infty}^{\infty} \frac{a}{\sqrt{a^2 + \sigma_d^2}} e^{-\frac{1}{2} \cdot \frac{(x-u_i)^2}{a^2 + \sigma_d^2}} \cdot e^{-x^2/W^2} dx
 \end{aligned}$$

Evaluating this integral leads to the following result.

$$P_{SSK|u_i} = \frac{1}{\sqrt{2}W_0} \cdot \frac{aW\sqrt{\pi}}{\sqrt{W^2 + 2(a^2 + \sigma_d^2)}} e^{-\frac{u_i^2}{W^2 + 2(a^2 + \sigma_d^2)}}$$

Considering the case where $n = 2$,

$$p_K|u_1, u_2 = 1 - \prod_{i=1}^2 \left(1 - \frac{1}{\sqrt{2}W_0} \frac{aW\sqrt{\pi}}{\sqrt{W^2+2(a^2+\sigma_d^2)}} e^{-\frac{u_i^2}{W^2+2(a^2+\sigma_d^2)}} \right)$$

Assuming that $u_1 = -u_2 = |u|$,

$$p_K|u_1, u_2 = p_K(u) = \frac{aW\sqrt{\pi}}{\sqrt{2}W_0\sqrt{W^2+2(a^2+\sigma_d^2)}} (2e^{-\frac{u^2}{W^2+2(a^2+\sigma_d^2)}} - \frac{aW\sqrt{\pi}}{\sqrt{2}W_0\sqrt{W^2+2(a^2+\sigma_d^2)}} e^{-\frac{2u^2}{W^2+2(a^2+\sigma_d^2)}})$$

$$p_K(u) = p_K(-u)$$

There comes up a question of what values of u give the maximum $p_K(u)$.

$$\frac{d}{du} p_K(u) = \frac{aW\sqrt{\pi}}{\sqrt{2}W_0\sqrt{W^2+2(a^2+\sigma_d^2)}} \left(\frac{-4u}{W^2+2(a^2+\sigma_d^2)} e^{-\frac{u^2}{W^2+2(a^2+\sigma_d^2)}} + \frac{4aW\sqrt{\pi}u}{\sqrt{2}W_0(W^2+2(a^2+\sigma_d^2))^{3/2}} e^{-\frac{2u^2}{W^2+2(a^2+\sigma_d^2)}} \right)$$

It is clear that $p_K(u)$ is $\begin{pmatrix} \text{increasing} \\ \text{extreme} \\ \text{decreasing} \end{pmatrix}$ at u , when

$p'_K(u) \begin{pmatrix} > \\ = \\ < \end{pmatrix} 0$; i.e., when

$$[W^2 + 2(a^2 + \sigma_d^2)] \ln \frac{aW\sqrt{\pi}}{\sqrt{2}W_0\sqrt{W^2 + 2(a^2 + \sigma_d^2)}} \begin{pmatrix} > \\ = \\ < \end{pmatrix} u^2$$

Let u increase from 0 to ∞ . If inside of the logarithm is less than one, then a relationship $<$ holds; i.e., $p_K(u)$ is always decreasing as u increases. In this case, it is not proposed to disperse aiming points. If the inside of the logarithm is greater than one, we have first $>$, $=$, and finally $<$. But this never happens because the inside of the logarithm is always less than one; i.e.,

$$\frac{aW\sqrt{\pi}}{\sqrt{2}W_0\sqrt{W^2 + 2(a^2 + \sigma_d^2)}} = \frac{a\sqrt{2}}{\sqrt{W^2 + 2(a^2 + \sigma_d^2)}} < 1$$

This result suggests one should aim at the origin instead of dispersing aiming points.

This suggestion contradicts the intuition that some dispersions of the aiming point might achieve greater kill of an area target. That is, if the model is considered in terms of cookie cutter lethality function, then it is clear that some dispersions, $|u|$, would do better than 2 rounds of impact on the same point (origin). This contradiction is mainly due to the assumption that cumulative damage is

negligible, that is, once killed elements can be killed again, and also the Gaussian lethality function.

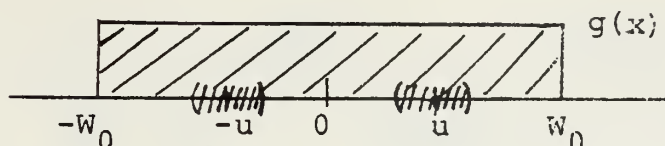


Fig. 10. Cookie Cutter Drops on Uniform Target

As a treatment to this problem, a suggestion is possible. The lethal effect of 2 rounds detonated at the same point will be approximately the same as that of one round, looked at from the mathematical point of view. Therefore, it is suggested that one round be aimed at the origin u_1 , and then the aiming points be dispersed such that u_2 and u_3 in Fig. 11 be the centers of the surviving area target, b to W_0 and a to $-W_0$, and so on.

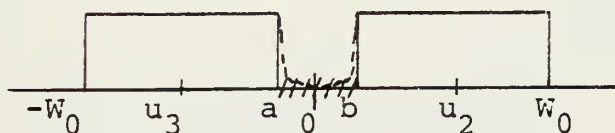


Fig. 11. Suggestion for Uniform Target

4. Normal Distribution of Target Elements

In this case, the distribution of the elements of an area target (linear) is as follows.

$$g(x) = \frac{1}{\sqrt{2\pi} \sigma_x} e^{-\frac{1}{2} \cdot \frac{x^2}{\sigma_x^2}}$$

$$\begin{aligned} P_{SSK|u_i} &= \frac{1}{\sqrt{2\pi} \sigma_x} \int_{-W_0}^{W_0} e^{-\frac{1}{2} \cdot \frac{x^2}{\sigma_x^2}} \cdot \frac{a}{\sqrt{a^2 + \sigma_d^2}} e^{-\frac{1}{2} \cdot \frac{(x-u_i)^2}{a^2 + \sigma_d^2}} dx \\ &= \frac{1}{\sqrt{2\pi} \sigma_x} \int_{-\infty}^{\infty} e^{-x^2/W^2} \cdot e^{-\frac{1}{2} \cdot \frac{x^2}{\sigma_x^2}} \cdot \frac{a}{\sqrt{a^2 + \sigma_d^2}} e^{-\frac{1}{2} \cdot \frac{(x-u_i)^2}{a^2 + \sigma_d^2}} dx \end{aligned}$$

Evaluating this integral leads to the following result.

$$P_{SSK|u_i} = \frac{aW}{\sqrt{(a^2 + \sigma_d^2)(2\sigma_x^2 + W^2) + W^2\sigma_x^2}} e^{-\frac{1}{2} \cdot \frac{(2\sigma_x^2 + W^2)u_i^2}{(a^2 + \sigma_d^2)(2\sigma_x^2 + W^2) + W^2\sigma_x^2}}$$

Considering the case where $n = 2$,

$$P_K|u_1, u_2 = 1 - \prod_{i=1}^2 \left(1 - \frac{aW}{\sqrt{(a^2 + \sigma_d^2)(2\sigma_x^2 + W^2) + W^2\sigma_x^2}} e^{-\frac{1}{2} \cdot \frac{(2\sigma_x^2 + W^2)u_i^2}{(a^2 + \sigma_d^2)(2\sigma_x^2 + W^2) + W^2\sigma_x^2}} \right)$$

assuming that $u_1 = -u_2 = |u|$,

$$p_K|u_1, u_2 = p_K(u) = \frac{aW}{\sqrt{(a^2 + \sigma_d^2)(2\sigma_x^2 + W^2) + W^2\sigma_x^2}}$$

$$\times (2e^{-\frac{1}{2} \cdot \frac{(2\sigma_x^2 + W^2)u^2}{(a^2 + \sigma_d^2)(2\sigma_x^2 + W^2) + W^2\sigma_x^2}} - \frac{aW}{\sqrt{(a^2 + \sigma_d^2)(2\sigma_x^2 + W^2) + W^2\sigma_x^2}} e^{-\frac{1}{2} \cdot \frac{2(\sigma_x^2 + W^2)u^2}{(a^2 + \sigma_d^2)(2\sigma_x^2 + W^2) + W^2\sigma_x^2}})$$

$$p_K(u) = p_K(-u)$$

There comes up a question of what values of u give the maximum $p_K(u)$.

$$\frac{d}{du} p_K(u) = \frac{aW}{\sqrt{(a^2 + \sigma_d^2)(2\sigma_x^2 + W^2) + W^2\sigma_x^2}}$$

$$\times \left(-\frac{2(2\sigma_x^2 + W^2)u}{(a^2 + \sigma_d^2)(2\sigma_x^2 + W^2) + W^2\sigma_x^2} e^{-\frac{1}{2} \cdot \frac{(2\sigma_x^2 + W^2)u^2}{(a^2 + \sigma_d^2)(2\sigma_x^2 + W^2) + W^2\sigma_x^2}} \right.$$

$$\left. + \frac{2aW(2\sigma_x^2 + W^2)u}{[(a^2 + \sigma_d^2)(2\sigma_x^2 + W^2) + W^2\sigma_x^2]^{3/2}} e^{-\frac{1}{2} \cdot \frac{2(2\sigma_x^2 + W^2)u^2}{(a^2 + \sigma_d^2)(2\sigma_x^2 + W^2) + W^2\sigma_x^2}} \right)$$

It is clear that $p_K(u)$ is $\begin{pmatrix} \text{increasing} \\ \text{extreme} \\ \text{decreasing} \end{pmatrix}$ at u , when

$$p'_K(u) \begin{pmatrix} > \\ = \\ < \end{pmatrix} 0; \text{ i.e.,}$$

$$\frac{2[(a^2 + \sigma_d^2)(2\sigma_x^2 + W^2) + W^2\sigma_x^2]}{2\sigma_x^2 + W^2} \ln \frac{aW}{\sqrt{(a^2 + \sigma_d^2)(2\sigma_x^2 + W^2) + W^2\sigma_x^2}} \begin{pmatrix} > \\ = \\ < \end{pmatrix} u^2$$

Let us vary from 0 to ∞ . If the inside of the logarithm is less than one, then a relationship $<$ holds; i.e., $p_K(u)$ is always decreasing as u increases. In this case, it is not proposed to disperse aiming points. If the inside of the logarithm is greater than one, we have first $<$, $=$, and finally $>$. But it never happens because

$$\frac{aW}{\sqrt{(a^2 + \sigma_d^2)(2\sigma_x^2 + W^2) + W^2\sigma_d^2}} < 1.$$

This result suggests one should aim at the origin instead of dispersing aiming points.

This suggestion contradicts with the intuition that some dispersions of aiming point might achieve greater kill of an area target. That is, if this model is considered in terms of cookie cutter lethality function, then it is clear that some dispersions, $|u|$, would do better than 2 rounds of impact on the same point (origin). This contradiction is mainly due to the assumption that cumulative damage is negligible and also the Gaussian lethality function. Probability distribution of elements may also contribute to this result of $|u|$ being close to the origin.

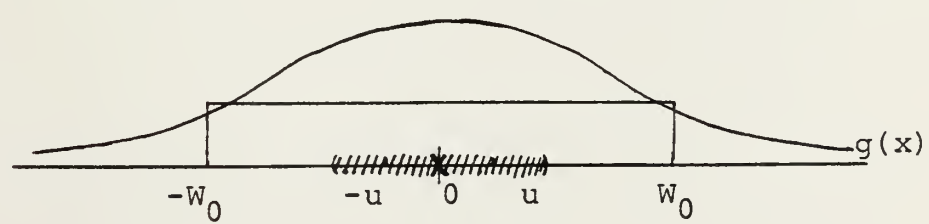


Fig. 12. Cookie Cutter Drops on Normal Target

As a treatment to this problem, a suggestion is possible. The lethal effect of 2 rounds detonated at the same point will be approximately the same with that of one round from the mathematical point of view, rather than from the real point of view. Therefore, it is suggested that one round be aimed at the origin (center of mass), u_1 , and then disperse the aiming points, u_2 and u_3 for the remaining elements as shown in Fig. 13.

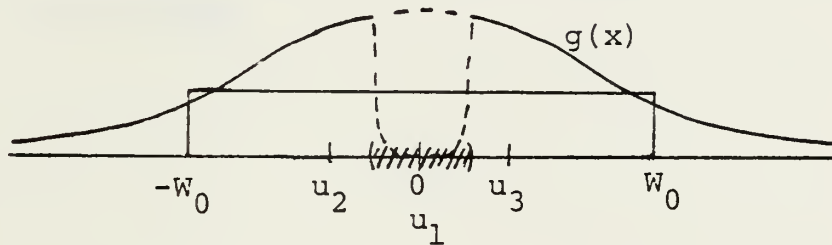


Fig. 13. Suggestion for Normal Target

VI. CONCLUSIONS

For a point target of known location, aiming at the center of mass achieves greater chance of a kill. For a point target of unknown location, adjusting or pattern fire would do better, but salvo fire is not effective. For an area target in which elements are uniformly or normally distributed, it is suggested to aim at the center of mass and thereafter to disperse the aiming points for the surviving elements of the target.

LIST OF REFERENCES

1. U.S. Army Material Development and Readiness Command, Army Weapon Systems Analysis, Part One, November 1977.
2. Taylor, James G., Mathematical Models of Combat, Naval Postgraduate School, February 1975.
3. von Neumann, John, Optimum Aiming at an Imperfectly Located Target, Appendix to Optimum Spacing of Bombs or Shots in the Presence of Systematic Errors, Ballistic Research Laboratory, Report 241, July 3, 1941.

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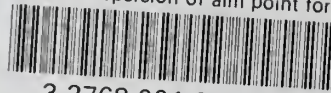
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